ρ-Metrics for Fixed Point Theory in Non-Metrizable Topologies

Muhammad Arshad*
Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan
marshadzia@iiu.edu.pk

We present a new framework for fixed point theory in non-metrizable spaces via ρ-metrics, maps

$$\rho\colon (X \times X) \setminus \Delta \to \mathbb{R},$$

which capture convergence through the condition $\rho(x_n, x) \to -\infty$. By constructing explicit ρ-metrics on the Sorgenfrey and Michael lines, we introduce forward and backward ρ-Cauchy completeness notions that resolve Suzuki's obstruction for rectangular metrics. Under these completeness hypotheses, Picard iterations of $\rho\psi$-contractions yield unique fixed points in the induced ρ-topology, even when classical continuity fails. Furthermore, we derive set-theoretic bounds on tightness and weight to quantify convergence thresholds, illustrating the adaptability of ρ-metrics for dynamics in non-uniformizable settings. As an application, we provide existence, uniqueness, and approximation of solutions for a class of nonlinear differential equations.

References

[1] Azam, A., Beg, I., & Arshad, M. (2010). Fixed point in topological vector space-valued cone metric spaces. *Fixed point theory and applications*, 2010(1), 604084. Suzuki, T. (2014).

[2] Beg, I., Azam, A., Arshad, M., & Latushkin, Y. (2009). Common fixed points for maps on topological vector space valued cone metric spaces. *Int. J. Math. Math. Sci., 2009, 560264-1.*

[3] Bella, A., & Spadaro, S. (2017). Cardinal Invariants for the G_δ topology. *arXiv preprint* arXiv:1707.04871.

[4] Bonanzinga, M. (1998). Star-Lindelof and absolutely star-Lindelof spaces. *Questions and Answers in General Topology*, 16, 79-104.

[5] Generalized metric spaces do not have the compatible topology. *In Abstract and applied Analysis* (Vol. 2014, No. 1, p. 458098). Hindawi Publishing Corporation.

[6] Secelean, N. A., & Wardowski, D. (2017). New fixed point tools in non-metrizable spaces. *Results in Mathematics*, 72(1), 919-935.