Recent Advances in Cardinal Inequalities

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Three fundamental results in the theory of cardinal functions were established in the middle of the last century. For a Hausdorff space X, $|X| \leq 2^{L(X)\chi(X)}$ (Arhangel'skii, 1969), $|X| \leq 2^{c(X)\chi(X)}$ (Hajnal-Juhász, 1967), and $|X| \leq 2^{hL(X)}$ (De Groot, 1965, and Smirnov, 1950).

Considerable progress has been made in recent years in improving these results and others in various directions. Unified proofs have been given as well as new reformulated proofs. Notably, Bella and Spadaro gave a unified improvement of Arhangel'skii's and Hajnal-Juhász' theorems. Carlson gave a bound for the density of any Hausdorff space that leads to a strict improvement of the Hajnal-Juhász theorem. Bella gave a bound for the cardinality of a regular space. Bella, Carlson, Spadaro, and Szeptycki recently modified this and gave a strict improvement of the De Groot-Smirnov theorem. Other results in the area have been given by Basile, Bella, Bonanzinga, Gotchev, Tkachenko, Tkachuk, and others.

After a broad survey of these developments, we present a new result. We show that $w(X) \leq hL(X)^{ot(X)}$ for any compact Hausdorff space X, where the o-tightness $ot(X)$ was defined by Tkachenko with the properties $ot(X) \leq t(X)$ and $ot(X) \leq c(X)$.

Key questions remain open. In 1979 Arhangel'skii asked, is the cardinality of a Hausdorff space X bounded by $2^{wL_c(X)\chi(X)}$? Recently Carlson has asked, is there a common proof of the three fundamental results mentioned above?

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