Infinite monochromatic configurations via ultrafilters and nonstandard analysis

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An area of Ramsey Theory studies problems of the following form: "Given a family $\mathcal{F} \subseteq \mathcal{P}(\mathbf{N})$, is it true that for every finite partition (coloring) $\mathbf{N} = C_1 \cup \ldots \cup C_r$ one can always find a "configuration" $F \in \mathcal{F}$ which is monochromatic, *i.e.* $F \subseteq C_i$ for some i?". If this is the case, one says that the family $\mathcal{F}$ is *partition regular* on $\mathbf{N}$ or – by abusing language – that the configurations of $\mathcal{F}$ are *monochromatic*.

Plenty of monochromatic configurations have been proved to exist. Historically, the first examples are provided by "Schur's triples" $\{x, y, x+y\}$ (Schur's Theorem, 1918), and ℓ-term arithmetic progressions $\{a, a+d, \ldots, a+(\ell-1)d\}$ for any fixed ℓ (van der Waerden's Theorem, 1927). In recent years, particular attention has been given to infinite configurations, the study of which seem to require somewhat different techniques. To this end, we have introduced the notion of *Ramsey partition regularity* [1], which is characterized in terms of ultrafilters belonging to the topological closure in $\beta\mathbb{N}^2$ of tensor powers $\mathcal{U} \otimes \mathcal{U}$, called *Ramsey's witnesses.* Using these characterizations, and working in a nonstandard setting where ultrafilters are represented as nonstandard natural numbers, we have proven several new results, both positive and negative, about the existence of infinite monochromatic configurations. Here are two examples:

- All and only families of infinite configurations of the form $\{a_n, c \cdot a_n^k + P(a_m) \mid n < m\}$ where (a_n) is an increasing sequence that are partition regular on $\mathbb{N}$, are the configurations $\{a_n, a_n + a_m \mid n < m\}$ and $\{a_n, a_m - a_n \mid n < m\}$.

  [This solves several open problems that were posed by Kra, Moreira, Richter, and Robertson [2]. For instance, both $\{a_n, 2a_n + a_m \mid n < m\}$ and $\{a_n, a_n + a_m^2 \mid n < m\}$ where (a_n) is an increasing sequence, are *not* partition regular on $\mathbf{N}$.]

- The family of infinite configurations of the form $\left\{\frac{a_i}{a_j}, \frac{a_i + a_j}{a_k} \mid i < j < k\right\}$ where (a_n) is increasing, is partition regular on $\mathbb{N}$. In particular, the family of configurations $\left\{b_n, b_n + b_m, \frac{b_n}{b_m} \mid n > m\right\}$ where (b_n) is increasing, is partition regular.

  [It is known that infinite configurations $\{b_n + b_m, b_n \cdot b_m \mid n > m\}$ are *not* partition regular (Hindman 1984). It is worth remarking that the monochromaticity of configurations $\{a, b, a+b, a \cdot b\}$ is still one of the main open problems in this area.]

References

[1] M. Di Nasso, L. Luperi Baglini, M. Mamino, R. Mennuni, and M. Ragosta, *Ramsey's witnesses*, arXiv:2503.09246, 2025.

[2] B. Kra, J. Moreira, F. Richter, and D. Robertson, *Problems on Infinite Sumset Configurations in the Integers and Beyond*, arXiv:2311.06197, 2023.

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