More Borel chromatic numbers of closed graphs

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We give an overview of recent consistency results on Borel chromatic numbers of closed graphs on Polish spaces. A graph G whose set of vertices is a Polish space X is *closed*, *Borel*, or *analytic* iff its edge relation is, as a subset of X^2 without the diagonal.

The *Borel chromatic number* of an analytic graph G on a Polish space X is the the least size of a family of *G-independent* Borel sets, i.e., Borel sets without edges, that covers all of X. Kechris, Solecki, and Todorčević [4] exhibited a closed graph G_0 on the Cantor space 2^ω with the following property: an analytic graph G on a Polish space has an uncountable Borel chromatic number iff there is a continuous graph homomorphism from G_0 to G. This *G_0-dichotomy* implies the classical dichotomies in descriptive set theory [5]. It follows that G_0 has the minimal uncountable Borel chromatic number among all analytic graphs on Polish spaces.

The upper end of the spectrum is less clear. Obviously, any graph that contains a complete subgraph, i.e., a *clique*, of size κ has chromatic number and hence Borel chromatic number at least κ. If a closed graph on a Polish space does not have a perfect clique, then its Borel chromatic number can be forced to be strictly below $2^{\aleph_0}$ [3]. There are uncountably Borel chromatic, closed graphs G_1 and E_0 on 2^ω whose Borel chromatic numbers can consistently be different from that of G_0 and from each other [2, 6, 7]. Each of these graphs is closely associated with a forcing notion and sometimes the graph theoretic point of view can give new information about classical forcing notions [1, 7].

References

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