

Ideal convergence and topologically torsion elements of the circle group

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Given a sequence $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ of integers, an element x of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is said to be *topologically \mathbf{u} -torsion* if the sequence $(u_n x)_{n \in \mathbb{N}}$ converges to 0 in \mathbb{T} . The subgroup of \mathbb{T} *characterized by \mathbf{u}* is defined as $t_{\mathbf{u}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \rightarrow 0\}$. The talk begins with a brief historical overview of topologically torsion elements and characterized subgroups of \mathbb{T} , highlighting their relevance in several areas of Mathematics, such as topological algebra, harmonic analysis and number theory, where the asymptotic behavior of sequences like $(u_n x)_{n \in \mathbb{N}}$ plays a key role.

Recently, various generalizations of these notions were introduced and studied by Dikranjan, Das, Ghosh, Bose, and He. These developments rely on weaker notions of convergence, the most general being \mathcal{I} -convergence, introduced by Cartan. For an ideal \mathcal{I} of \mathbb{N} , an element x of \mathbb{T} is *topologically $\mathbf{u}_{\mathcal{I}}$ -torsion* if the sequence $u_n x$ \mathcal{I} -converges to 0 in \mathbb{T} , that is, $\{n \in \mathbb{N} : u_n x \notin U\} \in \mathcal{I}$ for every neighborhood U of 0 in \mathbb{T} ; the corresponding subgroup of \mathbb{T} \mathcal{I} -characterized by \mathbf{u} is $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \xrightarrow{\mathcal{I}} 0\}$. For the ideal $\mathcal{F}in$ of all finite subsets of \mathbb{N} , $\mathcal{F}in$ -convergence is the usual convergence and $t_{\mathbf{u}}^{\mathcal{F}in}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$.

The second part of the talk presents recent results on \mathcal{I} -characterized subgroups of \mathbb{T} and their elements, that extend their counterparts for $\mathcal{I} = \mathcal{F}in$. In particular, we offer a complete description of the $\mathbf{u}_{\mathcal{I}}$ -torsion elements of \mathbb{T} for sequences of strictly positive integers \mathbf{u} such that u_n divides u_{n+1} for every $n \in \mathbb{N}$, under suitable hypotheses on the ideal \mathcal{I} .

*Based on joint works with D. Dikranjan, R. Di Santo, H. Weber (University of Udine).