Spectra of maximal almost orthogonal families of projections in the Calkin algebra

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Let H be an infinite dimensional separable complex Hilbert space with inner product $\langle\cdot|\cdot\rangle$. Let $\mathcal{B}(H)$ be a Banach space of bounded linear operators on H with the operator norm. In case when $H = \ell^2(\omega)$, we can distinguish a particular subalgebra of the Banach space $\mathcal{B}(H)$: we define $\mathcal{K}(H)$ as the smallest Banach subalgebra of $\mathcal{B}(H)$ containing all finite-dimensional operators, and we call its elements compact operators. So, $T \in \mathcal{B}(H)$ is compact if it is a limit of finite-rank operators.† The collection $\mathcal{K}(H)$ has the structure of a C*-algebra and is a ring-theoretical ideal in $\mathcal{B}(H)$.

The Calkin algebra is the quotient C*- algebra

$$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H),$$

where the quotient mapping is denoted by $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$. Every separable C*-algebra is isomorphic to a C*-subalgebra of the Calkin algebra. We are interested in the set of projections in the Calkin algebra, i.e., in the set:

$$P(\mathcal{C}(H)) = \{p \in \mathcal{C}(H) : p = p^* = p^2\}.$$

For a set $A \subseteq \omega$, let P_A be the projection onto $\ell^2(A) \subseteq \ell^2(\omega)$. The map $A \mapsto P_A$ embeds the Boolean algebra $\mathcal{P}(\omega)$ into the space of projections $P(H)$. The map $A \mapsto \pi(P_A)$ defines an embedding of $\mathcal{P}(\omega)/\operatorname{fin}$ into $P(\mathcal{C}(H))$. This map is called the diagonal embedding.

A family of projections $A \subseteq P(\mathcal{C}(H))$ is almost orthogonal if the product of any two elements $p, q \in A$ is the zero of the algebra $\mathcal{C}(H)$. In this paper we investigate the possible spectra of maximal almost orthogonal families of projections in the Calkin algebra.

The collection of projections $P(\mathcal{C}(H))$ is a natural object to study, as it can be identified with the lattice of projections on $\mathcal{B}(H)$ modulo a natural equivalence relation, so we can identify elements of $P(\mathcal{C}(H))$ with closed subspaces of $\mathcal{B}(H)$.

An important result by Wofsey is:

Theorem 1 (Wofsey, 2007) *Let A be a family of disjoint uncountable sets. Then*

$$\mathbb{P}_A \Vdash \forall X \in A\, \exists Y (|Y| = |X| \,\&\, Y \textit{ is a m.a.o.f. }).$$

In other words, for any family of cardinals C there is a forcing notion such that C is included in the spectrum of m.a.o.f.'s. Wofsey's result is an operator-theoretic counterpart of the (positive) result of Hechler concerning spectra of maximal almosts disjoint families of sets. We have been searching for an operator-theoretic counterpart of the (negative) strengthening of Hechler's result on spectra of mad families given by Blass.

Thus, our main question in this paper is: can we isolate conditions, under which a specific set of cardinals C can be not only included, but actually equal to the spectrum of maximal almost orthogonal family of projections in a given model of set theory?

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†Equivalently, an operator $T \in \mathcal{B}(H)$ is compact if the image of the closed unit ball $B \subset H$ under T is precompact, which in turn is equivalent to T being weak-norm continuous when restricted to B.

Theorem 2 *Assume GCH. Let C be a set of cardinals satisfying the following conditions:*

- $\forall \kappa \in C$ κ *is uncountable,*
- C *is closed,*
- $\forall \kappa \in [\aleph_1, |C|]$ $\kappa \in C$,
- $\forall \kappa \in C$ $\mathrm{cf}(\kappa) = \omega \Rightarrow \kappa^+ \in C$.

Then there exists a forcing notion $\mathbb{P}$ *such that it satisfies the countable chain condition and forces the spectrum of maximal almost orthogonal families to be exactly* C.

References

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