A New Construction Principle

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We use the framework of Abstract Elementary Classes (AECs) to introduce a new Construction Principle $\mathrm{CP}(\mathbf{K}, *)$, which strictly generalises the Construction Principle of Eklof, Mekler and Shelah from [1, 3] and allows for many novel applications beyond the setting of universal algebra. In particular, we argue that the undefinability of free algebras is an instance of a more general phenomenon, which occurs whenever we have a combination of *uncountable categoricity*, *canonical amalgamation* and a version of the *Construction Principle*. We thus prove the following:

Theorem. *Let* $(\mathbf{K}, \leqslant_*)$ *be an uncountably categorical canonical amalgamation class with* $\mathrm{LS}(\mathbf{K}, \leqslant_*) = \aleph_0$ *and with* $(\mathbf{K}, \leqslant_*)$*-universal models for all* $\kappa \geqslant \aleph_0$*. If* $(\mathbf{K}, \leqslant_*)$ *satisfies the Construction Principle* $\mathrm{CP}(\mathbf{K}, *)$*, then:*

1. *there is a* $\mathfrak{L}_{\infty,\omega_1}$*-free structure of size* $\aleph_1$ *not in* $\mathbf{K}$*;*
2. *if* $V = L$ *there is for all* $\kappa \geqslant \aleph_0$ *a* $\mathfrak{L}_{\infty,\kappa^+}$*-free structure of size* κ^+ *not in* $\mathbf{K}$*.*

Thus, it follows, in ZFC*, that* $\mathbf{K}$ *is not axiomatisable in* $\mathfrak{L}_{\infty,\omega_1}$ *and, under* $V = L$*, that* $\mathbf{K}$ *is not axiomatisable also in* $\mathfrak{L}_{\infty,\infty}$*.*

We then apply our general framework to show that $\mathrm{CP}(\mathbf{K}, *)$ holds in the classes of free products of cyclic groups of fixed order, direct sums of a fixed torsion-free abelian group of rank 1 which is not $\mathbb{Q}$, (infinite) free (k, n)-Steiner systems, and (infinite) free generalised n-gons. From this we derive, in ZFC, that these classes of structures are not axiomatisable in the infinitary logic $\mathfrak{L}_{\infty,\omega_1}$, and, under $V = L$, that they are not axiomatisable in $\mathfrak{L}_{\infty,\infty}$. This talk is based on the article preprint [2] and is joint work with Tapani Hyttinen and Gianluca Paolini.

References

[1] P. C. Eklof and A. H. Mekler. *Categoricity results for* $L_{\infty,\kappa}$. Ann. Pure Appl. Logic **37.1** (1988), 81-99.

[2] T. Hyttinen, G. Paolini and D. E. Quadrellaro. *A New Construction Principle.* arXiv:2505.10155.

[3] A. H. Mekler and S. Shelah. $L_{\infty,\omega}$*-free algebras.* Algebra Univers. **26** (1989), 351-366.

*This is joint work with Tapani Hyttinen (University of Helsinki) and Gianluca Paolini (University of Turin).